Purity filtration of multidimensional linear systems

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***Abstract*— In this paper, we show how the purity filtration of a finitely presented module, associated with a multidimensional linear system, can be explicitly characterized by means of classical concepts of module theory and homological algebra. Our approach avoids the use of sophisticated homological algebra methods such as spectral sequences used in [3], [4], [5], associated cohomology used in [9], and Spencer cohomology used in [12], [13]. It allows us to develop efficient implementations in the PURITYFILTRATION and `AbelianSystems` packages. The purity filtration gives an intrinsic classification of the torsion elements of the module by means of their grades, and thus a classification of the autonomous elements of the multidimensional linear system by means of their codimensions. The results developed here are used in [16] to determine an equivalent block-triangular linear system of the multidimensional linear system formed by equidimensional diagonal blocks. This equivalent linear system highly simplifies the computation of a Monge parametrization of the original linear system.**

I. ALGEBRAIC ANALYSIS APPROACH TO LINEAR SYSTEMS THEORY

In what follows, D will denote a *noetherian domain*, namely a ring without zero divisors (namely, $d_1\,d_2 = 0$ yields $d_1 = 0$ or $d_2 = 0$) such that every left (resp., right) ideal of D is finitely generated as a left (resp., right) D-module [19]. Moreover, $D^{q\times p}$ will denote the set of $q\times p$ matrices with entries in D and I_p the unit of $D^{p\times p}$.

Example 1.1: If k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and $k' = \mathbb{R}$ or $\mathbb{C}$, then $A_n(k)$ (resp., $B_n(k)$, $\hat{\mathcal{D}}_n(k)$, $\mathcal{D}_n(k')$) is the ring of partial differential (PD) operators in $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, with coefficients in the polynomial ring $k[x_1, \ldots, x_n]$ (resp., the ring of rational functions $k(x_1, \ldots, x_n)$, the ring of formal power series $k[\![x_1, \ldots, x_n]\!]$, the ring of locally convergent power series $k'\{x_1, \ldots, x_n\}$). These rings are noetherian domains [4].

If $R \in D^{q\times p}$ and $\mathcal{F}$ is a left D-module, then we can define the *linear system* or *behaviour*:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

The algebraic analysis approach to linear systems theory (see [6], [11], [14], [20], [21] and the references therein) is based on the following result due to Malgrange.

Theorem 1.1 ([10]): Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be the left D-module *finitely presented by the matrix* $R \in D^{q\times p}$, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M sending $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M, $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$ (i.e., f_j is the row vector of length p with 1 in j^{th} position and 0 elsewhere), $y_j = \pi(f_j)$ for $j = 1, \ldots, p$, $\mathcal{F}$ a left D-module, and $\hom_D(M, \mathcal{F})$ the abelian group defined by the left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$. Then, the abelian group homomorphism χ (i.e., $\mathbb{Z}$-linear map) defined by

$$\begin{array}{rcl} \chi : \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \phi & \longmapsto & \eta = (\phi(y_1)\ \ldots\ \phi(y_p))^T, \end{array}$$

is an isomorphism and its inverse χ^{-1} of χ is defined by

$$\begin{array}{rcl} \chi^{-1} : \ker_{\mathcal{F}}(R.) & \longrightarrow & \hom_D(M, \mathcal{F}) \\ \eta & \longmapsto & \phi_\eta, \end{array} \tag{1}$$

where ϕ_η is defined by $\phi_\eta(\pi(\lambda)) = \lambda\,\eta$ for all $\lambda \in D^{1\times p}$.

Theorem 1.1 proves that $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, where $\cong$ denotes an isomorphism [19]. Hence, the linear system $\ker_{\mathcal{F}}(R.)$ can be intrinsically studied by means of the two left D-modules M and $\mathcal{F}$. If f_j is the j^{th} vector of the standard basis of $D^{1\times p}$, then the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is finitely generated by $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$, namely $M = \sum_{j=1}^p D\,y_j$, where $\pi : D^{1\times p} \longrightarrow M$ is the left D-homomorphism which sends $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M. The generators y_j's of M satisfy the following relations

$$\sum_{j=1}^p R_{ij}\,y_j = \sum_{j=1}^p R_{ij}\,\pi(f_j) = \pi((R_{i1}\ \ldots\ R_{ip})) = 0,$$

for $i = 1, \ldots, q$. Let us now give the main idea of the proof of Theorem 1.1. If $\phi \in \hom_D(M, \mathcal{F})$ and $\eta_j = \phi(y_j)$ for $j = 1, \ldots, p$, then, for $i = 1, \ldots, q$, we get:

$$\sum_{j=1}^p R_{ij}\,\eta_j = \sum_{j=1}^p R_{ij}\,\phi(y_j) = \phi\left(\sum_{j=1}^p R_{ij}\,y_j\right) = \phi(0) = 0.$$

II. MODULE THEORY AND HOMOLOGICAL ALGEBRA

Since $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the properties of the left D-modules M and $\mathcal{F}$. Let us recall a few definitions.

Definition 2.1 ([19]): Let D be a left noetherian domain and M a finitely generated left D-module.

1) M is *free* if there exists $r \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $M \cong D^{1\times r}$. Then, r is called the *rank* of M.
2) M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.
3) M is *reflexive* if the left D-homomorphism

$$\begin{array}{rcl} \varepsilon : M & \longrightarrow & \hom_D(\hom_D(M, D), D), \\ m & \longmapsto & \varepsilon(m), \end{array}$$

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is an isomorphism, where:

$$\forall\, m \in M,\ \forall\, f \in \hom_D(M, D),\ \varepsilon(m)(f) = f(m).$$

4) M is *torsion-free* if the *torsion left D-submodule* of M, namely $t(M) = \{m \in M \mid \exists\, d \in D \backslash \{0\} :\ d\, m = 0\}$, is reduced to 0, i.e., $t(M) = 0$.
5) M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element.

Theorem 2.1 ([19]): A free module is projective, a projective is reflexive, and a reflexive is torsion-free.

Definition 2.2: 1) A *complex* of left D-modules

$$M_\bullet\ \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots, \tag{2}$$

is a sequence of left D-modules M_i and of left D-homomorphisms $d_i : M_i \longrightarrow M_{i-1}$ which satisfies:

$$\forall\, i \in \mathbb{Z}, \quad d_i \circ d_{i+1} = 0 \quad (\Leftrightarrow\ \operatorname{im} d_{i+1} \subseteq \ker d_i).$$

Similarly for a complex of right D-modules.

2) The *defect of exactness* of (2) at M_i is the left (resp., right) D-module defined by:

$$H_i(M_\bullet) \triangleq \ker d_i / \operatorname{im} d_{i+1}.$$

3) The complex (2) is *exact at* M_i if $H_i(M_\bullet) = 0$, i.e., if $\ker\ d_i = \operatorname{im} d_{i+1}$, and *exact* if $\ker\ d_i = \operatorname{im} d_{i+1}$ for all $i \in \mathbb{Z}$. An exact complex is called an *exact sequence*.
4) An exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \tag{3}$$

i.e., f is injective, $\ker g = \operatorname{im} f$ and g is surjective, is called a *short exact sequence*.

5) A *finite free resolution* of the left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{4}$$

where, for $i \geq 1$, $R_i \in D^{p_i \times p_{i-1}}$ and:

$$\begin{array}{rcl} .R_i : D^{1\times p_i} & \longrightarrow & D^{1\times p_{i-1}} \\ \lambda & \longmapsto & \lambda\, R_i. \end{array}$$

6) A *finite free resolution* of a right D-module N is an exact sequence of the form

$$0 \longleftarrow N \xleftarrow{\kappa} D^{q_0} \xleftarrow{S_1.} D^{q_1} \xleftarrow{S_2.} D^{q_2} \xleftarrow{S_3.} \ldots, \tag{5}$$

where, for $i \geq 1$, $S_i \in D^{q_{i-1} \times q_i}$ and:

$$\begin{array}{rcl} S_i. : D^{q_i} & \longrightarrow & D^{q_{i-1}} \\ \eta & \longmapsto & S_i\, \eta. \end{array}$$

Example 2.1: If D is a left noetherian domain and M a finitely generated left D-module, then we have the following short exact sequence of left D-modules:

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \tag{6}$$

Let $\mathcal{F}$ be a left D-module. Using (4), we can define the *extension abelian groups* $\operatorname{ext}^i_D(M, \mathcal{F})$'s for $i \geq 0$ as follows. Up to abelian group isomorphism, they are defined by the defects of exactness of the following complex

$$\begin{array}{ccccccc} \ldots & \xleftarrow{R_{i+1}.} & \mathcal{F}^{p_i} & \xleftarrow{R_i.} & \mathcal{F}^{p_{i-1}} & \xleftarrow{R_{i-1}.} & \ldots \\ \ldots & \xleftarrow{R_3.} & \mathcal{F}^{p_2} & \xleftarrow{R_2.} & \mathcal{F}^{p_1} & \xleftarrow{R_1.} & \mathcal{F}^{p_0} \longleftarrow 0, \end{array} \tag{7}$$

where $R_i. : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)(\eta) = R_i\, \eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$ and for all $i \geq 1$, namely:

$$\begin{cases} \operatorname{ext}^0_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\operatorname{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases}$$

The complex (7) is said to be obtained by application of the *contravariant left exact functor* $\hom_D(\,\cdot\,, \mathcal{F})$ to the *reduced (truncated) free resolution* of M, namely to the complex obtained by removing M from the finite free resolution (4):

$$\ldots \xrightarrow{.R_4} D^{1\times p_3} \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0.$$

A classical theorem of homological algebra proves that the $\operatorname{ext}^i_D(M, \mathcal{F})$'s depend only on the left D-modules M and $\mathcal{F}$ (up to abelian group isomorphism), i.e., they do not depend on the choice of the finite free resolution (4) of M [19].

Similarly, if D is a right noetherian ring, N a finitely generated right D-module, and $\mathcal{G}$ a right D-module, then, using the finite free resolution (5) of N, we can define:

$$\begin{cases} \operatorname{ext}^0_D(N, \mathcal{G}) = \hom_D(N, \mathcal{G}) \cong \ker_{\mathcal{G}}(.S_1), \\ \operatorname{ext}^i_D(N, \mathcal{G}) \cong \ker_{\mathcal{G}}(.S_{i+1})/\operatorname{im}_{\mathcal{G}}(.S_i), \quad i \geq 1. \end{cases}$$

Theorem 2.2 ([19]): Let (3) be a short exact sequence of left (resp., right) D-modules and $\mathcal{F}$ a left (resp., right) D-module. Then, the following long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{ext}^0_D(M'', \mathcal{F}) & \xrightarrow{g^\star} & \operatorname{ext}^0_D(M, \mathcal{F}) & \xrightarrow{f^\star} & \operatorname{ext}^0_D(M', \mathcal{F}) \\ & \xrightarrow{\kappa^1} & \operatorname{ext}^1_D(M'', \mathcal{F}) & \longrightarrow & \operatorname{ext}^1_D(M, \mathcal{F}) & \longrightarrow & \operatorname{ext}^1_D(M', \mathcal{F}) \\ & \xrightarrow{\kappa^2} & \operatorname{ext}^2_D(M'', \mathcal{F}) & \longrightarrow & \operatorname{ext}^2_D(M, \mathcal{F}) & \longrightarrow & \ldots, \end{array}$$

holds, where $f^\star$ (resp., $g^\star$) is defined by $f^\star(\phi) = \phi \circ f$ (resp., $g^\star(\psi) = \psi \circ g$) for all $\phi \in \hom_D(M, \mathcal{F})$ (resp., for all $\psi \in \hom_D(M'', \mathcal{F})$).

Proposition 2.1 ([19]): Let (3) be a short exact sequence of left (resp., right) D-modules and M a projective left (resp., right) D-module. Then, for every left (resp., right) D-module $\mathcal{F}$, we have $\operatorname{ext}^{i+1}_D(M'', \mathcal{F}) \cong \operatorname{ext}^i_D(M', \mathcal{F})$ for all $i \geq 1$.

If D is a ring, then we can define the concept of *left global dimension* $\operatorname{lgd}(D)$ (resp., *right global dimension* $\operatorname{rgd}(D)$) as the supremum of the minimal length of *projective resolutions* of left (resp., right) D-modules [19]. In what follows, we only need to know that they are two invariants of the ring D which coincide when D is a noetherian ring [19], and which is then simply denoted by $\operatorname{gld}(D)$.

Example 2.2: If k is a field, then we have $\operatorname{gld}(k[x_1, \ldots, x_n]) = n$ [19]. If k is a field of characteristic 0, $k' = \mathbb{R}$ or $\mathbb{C}$, and $D = A_n(k)$, $B_n(k)$, $\hat{\mathcal{D}}_n(k)$ or $\mathcal{D}_n(k')$, then $\operatorname{gld}(D) = n$ [4], [5], [9].

Theorem 2.3 ([1], [6], [9], [12], [14]): Let D be a noetherian domain having a finite global dimension $\mathrm{gld}(D) = n$, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by $R \in D^{q\times p}$, and the so-called *Auslander transpose* of M, namely the following right D-module:

$$N = D^q/(R\, D^p).$$

1) We have the following left D-isomorphism:
$$t(M) \cong \mathrm{ext}^1_D(N, D). \tag{8}$$
2) M is torsion-free iff $\mathrm{ext}^1_D(N, D) = 0$.
3) The following long exact sequence of left D-modules
$$0 \longrightarrow \mathrm{ext}^1_D(N, D) \longrightarrow M \xrightarrow{\varepsilon} \hom_D(\hom_D(M, D), D) \longrightarrow \mathrm{ext}^2_D(N, D) \longrightarrow 0 \tag{9}$$
holds, where ε is defined in 3 of Definition 2.1.
4) M is reflexive iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, 2$.
5) M is projective iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, n$.

Theorem 2.3 was implemented in the OREMODULES package [7] for some classes of noncommutative polynomial rings of *functional operators* (e.g., PD, shift, difference, time-delay operators) for which *Buchberger's algorithm* terminates for any admissible term order, and which computes a *Gröbner basis* [6]. Hence, using the OREMODULES package, we can effectively check whether or not a finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ admits torsion elements or is torsion-free, reflexive or projective.

Definition 2.3 ([19]): A left D-module $\mathcal{F}$ is *injective* if $\mathrm{ext}^i_D(M, \mathcal{F}) = 0$ for all left D-modules M and for all $i \geq 1$.

Example 2.3: If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{O}(\Omega)$) of smooth functions (resp., distributions/temperate distributions, real analytic/holomorphic functions) on Ω is an injective $D = k[\partial_1, \ldots, \partial_n]$-module ($k = \mathbb{R}$ or $\mathbb{C}$) [10], [11], [20].

If M is a left D-module admitting the finite free resolution

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

then, applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the previous exact sequence, and using the fact that $\mathrm{ext}^i_D(\,\cdot\,, \mathcal{F}) = 0$ for all $i \geq 1$, and Theorem 1.1, we obtain the following exact sequence of abelian groups:

$$\ldots \xleftarrow{.R_3} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

Hence, we get $\ker_{\mathcal{F}}(R_{i+1}.) = R_i\, \mathcal{F}^{p_{i-1}}$ for all $i \geq 1$. We then say that $\hom_D(\,\cdot\,, \mathcal{F})$ is an *exact* contravariant functor, i.e., transforms exact sequences of left D-modules into exact sequences of abelian groups.

Corollary 2.1 ([6], [14], [21]): Let D be a noetherian domain having a finite global dimension $\mathrm{gld}(D) = n$, $\mathcal{F}$ an injective left D-module, and $M = D^{1\times p}/(D^{1\times q}\, R)$ a left D-module finitely presented by $R \in D^{q\times p}$. If we set $Q_1 = R$, $p_1 = p$ and $p_0 = q$, then we have:

1) If M is a torsion-free left D-module, then there exists a matrix $Q_2 \in D^{p_1\times p_2}$ such that the following exact sequence of abelian groups
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2}$$
holds, i.e., $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}$. Then, Q_2 is called a *parametrization* of the linear system $\ker_{\mathcal{F}}(R.)$.
2) If M is a reflexive left D-module, then there exist $Q_2 \in D^{p_1\times p_2}$ and $Q_3 \in D^{p_2\times p_3}$ such that the following exact sequence of abelian groups
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2} \xleftarrow{Q_3.} \mathcal{F}^{p_3}$$
holds, i.e.:
$$\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}, \quad \ker_{\mathcal{F}}(Q_2.) = Q_3\, \mathcal{F}^{p_3}.$$
3) If M is a projective left D-module, then there exist n matrices $Q_i \in D^{p_{i-1}\times p_i}$ for all $i = 2, \ldots, n+1$ such that the following exact sequence
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \ldots \xleftarrow{Q_n.} \mathcal{F}^{p_n} \xleftarrow{Q_{n+1}.} \mathcal{F}^{p_{n+1}} \tag{10}$$
holds, i.e., $\ker_{\mathcal{F}}(Q_i.) = Q_{i+1}\, \mathcal{F}^{p_{i+1}}$ for $i = 1, \ldots, n$.

The matrices Q_i's defined in Theorem 2.1 can be computed by checking when the $\mathrm{ext}^i_D(N, D)$'s vanish [6]. For instance, applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the beginning of a finite free resolution $0 \longleftarrow N \xleftarrow{\kappa} D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$ of the Auslander transpose $N = D^q/(R\, D^p)$ of $M = D^{1\times p}/(D^{1\times q}\, R)$, we then get the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ and $\mathrm{ext}^1_D(N, D) \cong t(M) = \ker_D(.Q)/\mathrm{im}_D(.R)$. Hence, if $t(M) = 0$, then the above complex is exact at $D^{1\times p}$ and defines the beginning of a free resolution of the finitely presented left D-module $L = D^{1\times m}/(D^{1\times p}\, Q)$. Then, applying the exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to L, we obtain the exact sequence $\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$, which yields $\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$ and shows that the linear system $\ker_{\mathcal{F}}(R.)$ is parametrized by Q. Using the OREMODULES package [7], the matrices Q_i's of Corollary 2.1 can be effectively computed, which constructively solves the so-called *image representation problem of behaviours* (see [6], [14], [12], [20], [21] and the references therein). If $t(M) \neq 0$, then the autonomous elements of the linear system $\ker_{\mathcal{F}}(R.)$ correspond to the torsion elements of M, i.e., $t(M) = \ker_D(.Q)/\mathrm{im}_D(.R)$ [6], [14], [12], [20], [21]. If $R' \in D^{q'\times p}$ is a matrix such that $\ker_D(.Q) = D^{1\times q'}\, R'$, then $t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$, which shows that the residue class of the rows of R' in M defines a set of generators of the torsion left D-module $t(M)$ [6], [14].

Let us now introduce a useful lemma which gives a finite presentation of a quotient left D-module.

Proposition 2.2 ([8]): Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ be two matrices satisfying $D^{1\times q}\, R \subseteq D^{1\times q'}\, R'$, i.e., such that $R = R''\, R'$ for a certain $R'' \in D^{q\times q'}$. Moreover, let

$R_2' \in D^{r' \times q'}$ be a matrix such that $\ker_D(.R') = D^{1\times r'} R_2'$, and let π and π' be respectively the canonical projections:

$$\pi : D^{1\times q'} R' \longrightarrow (D^{1\times q'} R')/(D^{1\times q} R),$$
$$\pi' : D^{1\times q'} \longrightarrow D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} R_2').$$

Then, the left D-homomorphism ι defined by

$$\begin{array}{rcl} D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} R_2') & \xrightarrow{\iota} & (D^{1\times q'} R')/(D^{1\times q} R) \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\, R') \end{array} \quad (11)$$

is an isomorphism and its inverse ι^{-1} is defined by:

$$\begin{array}{rcl} (D^{1\times q'} R')/(D^{1\times q} R) & \xrightarrow{\iota^{-1}} & D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} R_2') \\ \pi(\lambda\, R') & \longmapsto & \pi'(\lambda). \end{array}$$

Applying Proposition 2.2 to the quotient left D-module $t(M) = (D^{1\times q'} R')/(D^{1\times q} R)$, we obtain

$$t(M) \cong D^{1\times q'} / \left(D^{1\times (q+r')} \, (R''^T \quad R_2'^T)^T\right), \qquad (12)$$

where $R'' \in D^{q\times q'}$ and $R_2' \in D^{r'\times q'}$ are defined by:

$$R = R''\, R', \quad \ker_D(.R') = D^{1\times r'} R_2'.$$

The *third isomorphism theorem* in module theory [19] yields:

$$\begin{aligned} M/t(M) &= [D^{1\times p}/(D^{1\times q} R)]/[(D^{1\times q'} R')/(D^{1\times q} R)] \\ &\cong D^{1\times p}/(D^{1\times q'} R'). \end{aligned} \qquad (13)$$

III. Monge parametrizations

According to 1 of Corollary 2.1, a linear system $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$ is parametrizable when the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q} R)$ is torsion-free and $\mathcal{F}$ is an injective left D-module. If M has torsion elements, i.e., $t(M) \neq 0$, and $\mathcal{F}$ is an injective left D-module, then applying the exact functor $\hom_D(\cdot, \mathcal{F})$ to the short exact sequence (6), we get the short exact sequence:

$$0 \longleftarrow \hom_D(t(M), \mathcal{F}) \xleftarrow{i^\star} \hom_D(M, \mathcal{F}) \xleftarrow{\rho^\star} \hom_D(M/t(M), \mathcal{F}) \longleftarrow 0.$$

We can then wonder if we can parametrize the linear system $\ker_{\mathcal{F}}(R.)$ by means of a more general parametrization than the one used in Corollary 2.1, i.e., by a parametrization obtained by glueing a parametrization of the parametrizable subsystem $\ker_{\mathcal{F}}(R'.) = Q'\,\mathcal{F}^m \cong \hom_D(M/t(M), \mathcal{F})$ of $\ker_{\mathcal{F}}(R.)$, where $M/t(M) = D^{1\times p}/(D^{1\times q'} R')$ (see (13)), with the integration of the (over)determined linear system $\ker_{\mathcal{F}}((R''^T \quad R_2'^T)^T.) \cong \hom_D(t(M), \mathcal{F})$ formed by the autonomous elements of $\ker_{\mathcal{F}}(R.)$ (see (12)). This leads us to the concept of a *Monge parametrization* [17], [18].

To recall the main results developed in [17], [18], let us first introduce a few more definitions [19].

Definition 3.1: 1) Let M and N be two left D-modules. An *extension of M by N* is a short exact sequence e of left D-modules of the form:

$$e:\, 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0. \qquad (14)$$

2) Two extensions of M by N

$$e_i:\, 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0, \quad i = 1, 2,$$

are said to be *equivalent*, denoted by $e_1 \sim e_2$, if there exists a left D-isomorphism $\phi : E_1 \longrightarrow E_2$ such that the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f_1} & E_1 & \xrightarrow{g_1} & M & \longrightarrow & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{f_2} & E_2 & \xrightarrow{g_2} & M & \longrightarrow & 0 \end{array}$$

holds, i.e., such that $f_2 = \phi \circ f_1$ and $g_1 = g_2 \circ \phi$.

3) Let $[e]$ be the equivalence class of the extension e for the equivalence relation $\sim$. The set of all equivalence classes of extensions of M by N is denoted by $\mathrm{e}_D(M, N)$.

Theorem 3.1 ([17], [18]): Let $M = D^{1\times p}/(D^{1\times q} R)$ and $N = D^{1\times s}/(D^{1\times t} S)$ be two finitely presented left D-modules, and $R_2 \in D^{r\times q}$ a matrix such that $\ker_D(.R) = D^{1\times r} R_2$. Then, every equivalence class of extensions of M by N is defined by the following extension of M by N

$$e:\, 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \qquad (15)$$

where the left D-module E is defined by the presentation

$$D^{1\times (q+t)} \xrightarrow{.Q} D^{1\times (p+s)} \xrightarrow{\varrho} E \longrightarrow 0, \qquad (16)$$

i.e., $E = D^{1\times (p+s)}/(D^{1\times (q+t)} Q)$, where

$$Q = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times (p+s)},$$

$$A \in \Omega = \{X \in D^{q\times s} \mid \exists\, Y \in D^{r\times t} : R_2\, X = Y\, S\}, \quad (17)$$

$$\begin{array}{rcl} N & \xrightarrow{\alpha} & E \\ \delta(\mu) & \longmapsto & \varrho(\mu\,(0 \quad I_s)), \end{array} \qquad \begin{array}{rcl} E & \xrightarrow{\beta} & M \\ \varrho(\lambda) & \longmapsto & \pi(\lambda\,(I_p \quad 0)^T), \end{array}$$

where $\pi : D^{1\times p} \longrightarrow M$ (resp., $\delta : D^{1\times s} \longrightarrow N$, and $\varrho : D^{1\times (p+s)} \longrightarrow E$) is the canonical projection onto M (resp., N, E). Finally, the equivalence class $[e]$ depends only on the residue class $\epsilon(A)$ of the matrix A in:

$$\Omega/(R\, D^{p\times s} + D^{q\times t}\, S) \cong \mathrm{ext}^1_D(M, N). \qquad (18)$$

The next corollary of Theorem 3.1 explains how to determine $\epsilon(A)$ for a given extension of M by N.

Corollary 3.1 ([18]): With the notations of Theorem 3.1, if we consider the following extension

$$0 \longrightarrow N \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0 \qquad (19)$$

of $M = D^{1\times p}/(D^{1\times q} R)$ by $N = D^{1\times s}/(D^{1\times t} S)$, and if $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$, $y_j = \pi(f_j)$ for all $j = 1, \ldots, p$, $z_j \in F$ a pre-image of y_j under v, then $\sum_{j=1}^p R_{ij}\, z_j \in \mathrm{im}\, u$ for all $i = 1, \ldots, q$, and since u is injective, there exists a unique $n_i \in N$ satisfying:

$$u(n_i) = \sum_{j=1}^p R_{ij}\, z_j.$$

If we consider a pre-image $a_i \in D^{1\times s}$ of n_i under δ, i.e., $n_i = \delta(a_i)$ for all $i = 1, \ldots, q$, then the extension (19) of M by N belongs to the same equivalence class of (15), where the left D-module E is defined by (16) with:

$$A = (a_1 \ \ldots \ a_q)^T \in D^{q\times s}.$$

Equivalently, the following commutative exact diagram

$$\begin{array}{cccccccc} & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \psi & & \| & \\ 0 \longrightarrow & & N & \xrightarrow{u} & F & \xrightarrow{v} & M & \longrightarrow 0 \end{array}$$

holds, where ψ and ϕ are respectively defined by

$$\begin{array}{rcl} \psi : D^{1\times p} & \longrightarrow & F \\ f_j & \longmapsto & z_j, \ j = 1, \ldots, p, \\ \phi : D^{1\times q} & \longrightarrow & N \\ e_i & \longmapsto & n_i = \delta(a_i), \ i = 1, \ldots, q, \end{array}$$

and $\{e_i\}_{i=1,\ldots,q}$ is the standard basis of $D^{1\times q}$.

Corollary 3.2 ([17], [18]): With the previous notations, an extension of $M/t(M)$ by $t(M)$, namely

$$e : 0 \longrightarrow t(M) \xrightarrow{\alpha} E \xrightarrow{\beta} M/t(M) \longrightarrow 0, \qquad (20)$$

can be defined by the finitely presented left D-module

$$E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\, P),$$

where the matrix P is given by

$$P = \begin{pmatrix} R' & -A \\ 0 & R'' \\ 0 & R'_2 \end{pmatrix} \in D^{(q'+q+r')\times(p+q')}, \qquad (21)$$

and the matrix A belongs to the abelian group Ω defined by:

$$\Omega = \left\{ A \in D^{q'\times q'} \mid \exists\, B \in D^{r'\times(q+r')} : \ R'_2\, A = B \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right\}. \qquad (22)$$

Moreover, the equivalence classes of the extensions of $M/t(M)$ by $t(M)$ depend only on the residue classes $\epsilon(A)$ of the matrix $A \in \Omega$ in the following abelian group:

$$\Omega / \left(R'\, D^{p\times q'} + D^{q'\times(q+r')} \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right) \qquad (23)$$
$$\cong \mathrm{ext}^1_D(M/t(M), t(M))).$$

More precisely, we have the following important results.

Theorem 3.2 ([17], [18]): Let $R \in D^{q\times p}$, $R' \in D^{q'\times p}$, $R'' \in D^{q\times q'}$, and $R'_2 \in D^{r'\times q'}$ be four matrices satisfying $M = D^{1\times p}/(D^{1\times q} R)$, $M/t(M) = D^{1\times p}/(D^{1\times q'} R')$, $R = R''\, R'$, and $\ker_D(.R') = D^{1\times r'}\, R'_2$. Moreover, let $E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\, P)$ be the left D-module finitely presented by the matrix P defined by

$$P = \begin{pmatrix} R' & -I_{q'} \\ 0 & R'' \\ 0 & R'_2 \end{pmatrix} \in D^{(q'+q+r')\times(p+q')}, \qquad (24)$$

and $\varrho : D^{1\times(p+q')} \longrightarrow E$ (resp., $\pi : D^{1\times p} \longrightarrow M$) the canonical projection onto E (resp., M). Then, we have:

1) $M \cong E$ for the following left D-isomorphism:

$$\begin{array}{rcl} M & \longrightarrow & E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\, P) \\ \pi(\lambda) & \longmapsto & \varrho(\lambda U), \quad U = (I_p \quad 0) \in D^{p\times(p+q')}. \end{array}$$

2) The following two extensions of $M/t(M)$ by $t(M)$

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0,$$
$$0 \longrightarrow t(M) \xrightarrow{\alpha} E \xrightarrow{\beta} M/t(M) \longrightarrow 0,$$

belong to the same equivalence class in:

$$e_D(M/t(M), t(M)).$$

3) For every left D-module $\mathcal{F}$, $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(P.)$,

$$\text{i.e.,} \quad R\,\eta = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} R'\,\zeta - \theta = 0, \\ R''\,\theta = 0, \\ R'_2\,\theta = 0, \end{array} \right. \qquad (25)$$

for the following invertible transformations:

$$\begin{array}{rcl} \gamma : \ker_{\mathcal{F}}(P.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \begin{pmatrix} \zeta \\ \theta \end{pmatrix} & \longmapsto & \eta = U \begin{pmatrix} \zeta \\ \theta \end{pmatrix} = \zeta, \\ \gamma^{-1} : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(P.) \\ \eta & \longmapsto & \begin{pmatrix} \zeta \\ \theta \end{pmatrix} = \begin{pmatrix} I_p \\ R' \end{pmatrix} \eta. \end{array}$$

IV. Purity filtration

To integrate the (under)determined linear system $\ker_{\mathcal{F}}(R.)$, (25) shows that we first have to integrate the following (over)determined linear system:

$$\ker_{\mathcal{F}}\left((R''^T \quad R'^T_2)^T.\right) \cong \hom_D(t(M), \mathcal{F}). \qquad (26)$$

Generalizing the ideas developed in Sections II and III, the goal of this section is to show how we can "zoom" in (26) and write (26) as a block-triangular linear system whose diagonal-blocks define equidimensional linear systems. To do that, we first need to introduce the concept of *purity filtration* of the left D-module M [4], [5], [9]. In [16], these results are used to obtain an equivalent block-triangular linear system, which is extremely useful for the computation of a Monge parametrization of the linear system $\ker_{\mathcal{F}}(R.)$. For more details, see [16]. The results developed in this section generalize the ones obtained in [15] for linear 2D systems.

Let M be a finitely generated left D-module. We can consider the beginning of a finite free resolution of M:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \xleftarrow{.R_3} D^{1\times p_3}.$$

Then, the defects of exactness of the following complex

$$0 \longrightarrow D^{p_0} \xrightarrow{R_1 .} D^{p_1} \xrightarrow{R_2 .} D^{p_2} \xrightarrow{R_3 .} D^{p_3} \qquad (27)$$

are the right D-modules defined by:

$$\begin{cases} \mathrm{ext}^2_D(M,D) \cong \ker_D(R_3.)/\mathrm{im}_D(R_2.), \\ \mathrm{ext}^1_D(M,D) \cong \ker_D(R_2.)/\mathrm{im}(R_1.), \\ \mathrm{ext}^0_D(M,D) \cong \ker_D(R_1.). \end{cases} \tag{28}$$

To characterize the right D-modules $\mathrm{ext}^i_D(M,D)$'s, we need to compare $\ker_D(R_i.)$ with $\mathrm{im}_D(R_{i-1}.) = R_{i-1}\, D^{p_{i-2}}$. For a fixed k from 1 to 3, let us introduce the notations $R_{kk} = R_k$, $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1}$, and:

$$N_{kk} = \mathrm{coker}_D(R_{kk}.) = D^{p_{kk}}/(R_{kk}\, D^{p_{(k-1)k}}).$$

Then, for $1 \leq k \leq 3$, let us consider the beginning of a finite free resolution of the right D-module N_{kk}:

$$\ldots \xrightarrow{R_{(k-1)k}.} D^{p_{(k-1)k}} \xrightarrow{R_{kk}.} D^{p_{kk}} \xrightarrow{\kappa_{kk}} N_{kk} \longrightarrow 0. \tag{29}$$

The choice of notations is natural: if we write the 3 long exact sequences (29) for $k = 1, 2, 3$ on the same page, where the k^{th} exact sequence of (29) is written at level k as shown in Fig. 1, then the free right D-module $D^{p_{jk}}$ is at position (j,k) and R_{jk} arrives at $D^{p_{jk}}$ with $j \leq k$.

Since (27) is a complex of right D-modules, we obtain $R_{kk}\, R_{(k-1)(k-1)} = R_k\, R_{k-1} = 0$ for all $k = 2, 3$, and thus

$$R_{(k-1)(k-1)}\, D^{p_{(k-2)(k-1)}} \subseteq \ker_D(R_{kk}.) = R_{(k-1)k}\, D^{p_{(k-2)k}},$$

i.e., matrices $F_{(k-2)k} \in D^{p_{(k-2)k} \times p_{(k-2)(k-1)}}$ exist such that:

$$\forall\, k = 1, 2, 3, \quad R_{(k-1)(k-1)} = R_{(k-1)k}\, F_{(k-2)k}. \tag{30}$$

Using (30) for $k = 1, 2$, we obtain

$$R_{(k-1)k}\, F_{(k-2)k}\, R_{(k-2)(k-1)} = R_{(k-1)(k-1)}\, R_{(k-2)(k-1)} = 0,$$

which yields

$$\begin{aligned} F_{(k-2)k}\ R_{(k-2)(k-1)}\ D^{p_{(k-3)(k-1)}} &\subseteq \ker_D(R_{(k-1)k}.) \\ &= R_{(k-2)k} D^{p_{(k-3)k}}, \end{aligned}$$

and there exists $F_{(k-3)k} \in D^{p_{(k-3)k} \times p_{(k-3)(k-1)}}$ such that:

$$F_{(k-2)k}\ R_{(k-2)(k-1)} = R_{(k-2)k}\, F_{(k-3)k}. \tag{31}$$

Similarly, for $k = 3$, there exists $F_{-13} \in D^{p_{-13} \times p_{-12}}$ such that $F_{03}\, R_{02} = R_{03}\, F_{-13}$. Therefore, we obtain the commutative diagram of right D-modules (33) whose horizontal sequences are exact and where:

$$\begin{aligned} R_{00} = 0, \quad N_{00} = D^{p_{00}}/0 \cong D^{p_{00}}, \\ p_{00} = p_{01}, \quad p_{12} = p_{11}, \quad p_{23} = p_{22}. \end{aligned} \tag{32}$$

If we denote by N_{jk} the right D-module defined by

$$N_{jk} = \mathrm{coker}_D(R_{jk}.) = D^{p_{jk}}/(R_{jk}\, D^{p_{(j-1)k}}),$$

then, using (33), we obtain the commutative diagram (34) whose horizontal sequences are exact. Moreover, we have the following short exact sequences:

$$\begin{aligned} 0 \longrightarrow N_{13} \longrightarrow D^{p_{23}} \longrightarrow N_{23} \longrightarrow 0, \\ 0 \longrightarrow N_{23} \longrightarrow D^{p_{33}} \longrightarrow N_{33} \longrightarrow 0, \\ 0 \longrightarrow N_{12} \longrightarrow D^{p_{22}} \longrightarrow N_{22} \longrightarrow 0, \\ 0 \longrightarrow N_{01} \longrightarrow D^{p_{11}} \longrightarrow N_{11} \longrightarrow 0. \end{aligned} \tag{35}$$

Now, using (28), we obtain the following characterization of right D-modules $\mathrm{ext}^i_D(M,D)$'s:

$$\begin{cases} \mathrm{ext}^2_D(M,D) \cong \ker_D(R_{33}.)/\mathrm{im}_D(R_{22}.) \\ \qquad\qquad\qquad = (R_{23}\, D^{p_{13}})/(R_{22}\, D^{p_{12}}), \\ \mathrm{ext}^1_D(M,D) \cong \ker_D(R_{22}.)/\mathrm{im}_D(R_{11}.) \\ \qquad\qquad\qquad = (R_{12}\, D^{p_{02}})/(R_{11}\, D^{p_{01}}), \\ \mathrm{ext}^0_D(M,D) \cong \ker_D(R_{11}.)/\mathrm{im}_D(R_{00}.) = R_{01}\, D^{p_{-11}}. \end{cases} \tag{36}$$

Then, using (32), (36) yields the three short exact sequences (37) of right D-modules. Now, applying the contravariant exact functor $\hom_D(\,\cdot\,, D)$ to the three short exact sequences of (37) and using Theorem 2.2, we obtain the long exact sequences shown in Fig. 4.

In what follows, we shall suppose that the ring D satisfies

$$\forall\, i \in \mathbb{N}, \quad \mathrm{ext}^i_D(\mathrm{ext}^{i+1}_D(M,D), D) = 0, \tag{38}$$

for all left D-modules M. For instance, this condition holds if D is an *Auslander regular ring* [5], namely a noetherian ring with a finite global dimension $\mathrm{gld}(D)$ such as, for every $i \in \mathbb{N}$ and for every finitely generated left D-module M, any left D-submodule P of $\mathrm{ext}^i_D(M,D)$ satisfies $j_D(P) \geq i$, where the *grade* $j_D(P)$ of P [4], [5] is defined by:

$$j_D(P) = \min\{i \geq 0 \mid \mathrm{ext}^i_D(P,D) \neq 0\}. \tag{39}$$

For instance, the rings $k[x_1, \ldots, x_n]$, $A_n(k)$, $B_n(k)$, $\hat{\mathcal{D}}_n(k)$ and $\mathcal{D}_n(k')$ defined in Example 1.1 are Auslander regular [4], [5]. In particular, using (39), we get:

$$\mathrm{ext}^0_D(\mathrm{ext}^1_D(M,D),D) = 0, \ \ \mathrm{ext}^1_D(\mathrm{ext}^2_D(M,D),D) = 0.$$

Moreover, $\mathrm{ext}^1_D(N_{00}, D)$ is equal to 0 since $N_{00} = D^{p_{00}}$ is a free, and thus a projective right D-module (see, e.g., Corollary 6.58 of [19]). Therefore, the three long exact sequences in Fig. 4 yield the exact sequences (40). Using Proposition 2.1, the short exact sequences of (35) then yield:

$$\begin{cases} \mathrm{ext}^3_D(N_{33},D) \cong \mathrm{ext}^2_D(N_{23},D) \cong \mathrm{ext}^1_D(N_{13},D), \\ \mathrm{ext}^2_D(N_{22},D) \cong \mathrm{ext}^1_D(N_{12},D), \\ \mathrm{ext}^2_D(N_{11},D) \cong \mathrm{ext}^1_D(N_{01},D). \end{cases}$$

Since $N_{11} = D^{p_{11}}/(R_{11}\, D^{p_{01}})$ is the Auslander transpose of $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$, 1 of Theorem 2.3 gives:

$$t(M) \cong \mathrm{ext}^1_D(N_{11}, D).$$

A right D-module analogue of Theorem 1.1 shows that $\mathrm{ext}^0_D(N_{01}, D) \cong \ker_D(.R_{01})$, and (6) yields:

$$M/t(M) = D^{1\times p_{00}}/\ker_D(.R_{01}).$$

(40) yields the three exact sequences (41). Combining the long exact sequences (41) with the long exact sequence (9) and using $\mathrm{coker}\,\varepsilon = M/t(M)$ (see 3 of Theorem 2.3), we obtain the exact diagram (42), where:

$$\begin{aligned} \mathrm{coker}\,\gamma_{32} &\cong \mathrm{im}\,\gamma_{22} \subseteq \mathrm{ext}^2_D(\mathrm{ext}^2_D(M,D),D), \\ \mathrm{coker}\,\gamma_{21} &\cong \mathrm{im}\,\gamma_{11} \subseteq \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D), \\ \mathrm{coker}\, i &= M/t(M) \cong \mathrm{coker}\,\gamma_{10} \\ &\cong \mathrm{im}\,\gamma_{00} \subseteq \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D). \end{aligned} \tag{43}$$

$$\begin{array}{cccccccccccc}
D^{p_{-13}} & \xrightarrow{R_{03}\cdot} & D^{p_{03}} & \xrightarrow{R_{13}\cdot} & D^{p_{13}} & \xrightarrow{R_{23}\cdot} & D^{p_{23}} & \xrightarrow{R_{33}\cdot} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow 0, \\
D^{p_{-12}} & \xrightarrow{R_{02}\cdot} & D^{p_{02}} & \xrightarrow{R_{12}\cdot} & D^{p_{12}} & \xrightarrow{R_{22}\cdot} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0, & \\
D^{p_{-11}} & \xrightarrow{R_{01}\cdot} & D^{p_{01}} & \xrightarrow{R_{11}\cdot} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0. & & &
\end{array}$$

Fig. 1. Free resolutions of the N_{kk}'s

$$\begin{array}{ccccccccccccc}
D^{p_{-13}} & \xrightarrow{R_{03}\cdot} & D^{p_{03}} & \xrightarrow{R_{13}\cdot} & D^{p_{13}} & \xrightarrow{R_{23}\cdot} & D^{p_{23}} & \xrightarrow{R_{33}\cdot} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow & 0 \\
\uparrow F_{-13}\cdot & & \uparrow F_{03}\cdot & & \uparrow F_{13}\cdot & & \| & & & & & & \\
D^{p_{-12}} & \xrightarrow{R_{02}\cdot} & D^{p_{02}} & \xrightarrow{R_{12}\cdot} & D^{p_{12}} & \xrightarrow{R_{22}\cdot} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0 & & \\
\uparrow F_{-12}\cdot & & \uparrow F_{02}\cdot & & \| & & & & & & & & \\
D^{p_{-11}} & \xrightarrow{R_{01}\cdot} & D^{p_{01}} & \xrightarrow{R_{11}\cdot} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0 & & & & \\
\uparrow & & \| & & & & & & & & & & \\
0 & \longrightarrow & D^{p_{00}} & \xrightarrow{\kappa_{00}} & N_{00} & \longrightarrow & 0. & & & & & &
\end{array} \tag{33}$$

Fig. 2. Commutative diagram with horizontal exact sequences

$$\begin{array}{ccccccccc}
D^{p_{-13}} & \xrightarrow{R_{03}\cdot} & D^{p_{03}} & \xrightarrow{R_{13}\cdot} & D^{p_{13}} & \xrightarrow{\kappa_{13}} & N_{13} & \longrightarrow & 0 \\
\uparrow F_{-13}\cdot & & \uparrow F_{03}\cdot & & \uparrow F_{13}\cdot & & & & \\
D^{p_{-12}} & \xrightarrow{R_{02}\cdot} & D^{p_{02}} & \xrightarrow{R_{12}\cdot} & D^{p_{12}} & \xrightarrow{\kappa_{12}} & N_{12} & \longrightarrow & 0 \\
\uparrow F_{-12}\cdot & & \uparrow F_{02}\cdot & & \| & & & & \\
D^{p_{-11}} & \xrightarrow{R_{01}\cdot} & D^{p_{01}} & \xrightarrow{R_{11}\cdot} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0.
\end{array} \tag{34}$$

Fig. 3. Commutative diagram with horizontal exact sequences

$$\begin{array}{c}
0 \longrightarrow \mathrm{ext}^2_D(M,D) \longrightarrow N_{22} = D^{p_{23}}/(R_{22}\,D^{p_{12}}) \longrightarrow N_{23} = D^{p_{23}}/(R_{23}\,D^{p_{13}}) \longrightarrow 0, \\
0 \longrightarrow \mathrm{ext}^1_D(M,D) \longrightarrow N_{11} = D^{p_{12}}/(R_{11}\,D^{p_{01}}) \longrightarrow N_{12} = D^{p_{12}}/(R_{12}\,D^{p_{02}}) \longrightarrow 0, \\
0 \longrightarrow \mathrm{ext}^0_D(M,D) \longrightarrow N_{00} = D^{p_{00}} \longrightarrow N_{01} = D^{p_{01}}/(R_{01}\,D^{p_{01}}) \longrightarrow 0.
\end{array} \tag{37}$$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{ext}^0_D(N_{23},D) & \longrightarrow & \mathrm{ext}^0_D(N_{22},D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^2_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^1_D(N_{23},D) & \longrightarrow & \mathrm{ext}^1_D(N_{22},D) & \longrightarrow & \mathrm{ext}^1_D(\mathrm{ext}^2_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^2_D(N_{23},D) & \longrightarrow & \mathrm{ext}^2_D(N_{22},D) & \longrightarrow & \mathrm{ext}^2_D(\mathrm{ext}^2_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^3_D(N_{23},D) & \longrightarrow & \mathrm{ext}^3_D(N_{22},D) & \longrightarrow & \ldots \\
0 & \longrightarrow & \mathrm{ext}^0_D(N_{12},D) & \longrightarrow & \mathrm{ext}^0_D(N_{11},D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^1_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^1_D(N_{12},D) & \longrightarrow & \mathrm{ext}^1_D(N_{11},D) & \longrightarrow & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^2_D(N_{12},D) & \longrightarrow & \mathrm{ext}^2_D(N_{11},D) & \longrightarrow & \ldots \\
0 & \longrightarrow & \mathrm{ext}^0_D(N_{01},D) & \longrightarrow & \mathrm{ext}^0_D(N_{00},D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) \\
 & \longrightarrow & \mathrm{ext}^1_D(N_{01},D) & \longrightarrow & \mathrm{ext}^1_D(N_{00},D). & &
\end{array}$$

Fig. 4. Long exact sequences

$$\begin{array}{lcccccccc}
0 \longrightarrow & \mathrm{ext}^2_D(N_{23},D) & \longrightarrow & \mathrm{ext}^2_D(N_{22},D) & \longrightarrow & \mathrm{ext}^2_D(\mathrm{ext}^2_D(M,D),D), & & & \\
0 \longrightarrow & \mathrm{ext}^1_D(N_{12},D) & \longrightarrow & \mathrm{ext}^1_D(N_{11},D) & \longrightarrow & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D), & & & \\
0 \longrightarrow & \mathrm{ext}^0_D(N_{01},D) & \longrightarrow & \mathrm{ext}^0_D(N_{00},D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) & \longrightarrow & \mathrm{ext}^1_D(N_{01},D) & \longrightarrow 0.
\end{array} \tag{40}$$

$$\begin{array}{lcccccccc}
0 \longrightarrow & \mathrm{ext}^3_D(N_{33},D) & \xrightarrow{\gamma_{32}} & \mathrm{ext}^2_D(N_{22},D) & \xrightarrow{\gamma_{22}} & \mathrm{ext}^2_D(\mathrm{ext}^2_D(M,D),D) & \longrightarrow & \mathrm{coker}\,\gamma_{22} & \longrightarrow 0, \\
0 \longrightarrow & \mathrm{ext}^2_D(N_{22},D) & \xrightarrow{\gamma_{21}} & t(M) & \xrightarrow{\gamma_{11}} & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D) & \longrightarrow & \mathrm{coker}\,\gamma_{11} & \longrightarrow 0, \\
0 \longrightarrow & \mathrm{ext}^0_D(N_{01},D) & \xrightarrow{\gamma_{10}} & D^{1\times p_{00}} & \xrightarrow{\gamma_{00}} & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) & \longrightarrow & \mathrm{ext}^2_D(N_{11},D) & \longrightarrow 0.
\end{array} \tag{41}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathrm{ext}_D^3(N_{33}, D) \arrow[r, "\gamma_{32}"] & \mathrm{ext}_D^2(N_{22}, D) \arrow[r] \arrow[d, "\gamma_{21}"] & \mathrm{coker}\,\gamma_{32} \arrow[r] & 0 & \\
 & 0 \arrow[r] & t(M) \arrow[r, "i"] \arrow[d] & M \arrow[r, "\rho"] & M/t(M) \arrow[r] & 0. \\
 & & \mathrm{coker}\,\gamma_{21} \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd} \tag{42}$$

We then get the following filtration $\{M_i\}_{i=0,\ldots,3}$ of M:

$$\begin{array}{c} 0 \subseteq M_3 = (\gamma_{21} \circ \gamma_{32})(\mathrm{ext}_D^3(N_{33}, D)) \\ \subseteq M_2 = \gamma_{21}(\mathrm{ext}_D^2(N_{22}, D)) \subseteq M_1 = t(M) \subseteq M_0 = M. \end{array} \tag{44}$$

Definition 4.1 ([4], [5]): A finitely generated left D-module M is called *pure* or $j_D(M)$*-pure* if $j_D(P) = j_D(M)$ for all non-zero left D-submodules P of M.

Theorem 4.1 ([4], [5]): Let D an Auslander regular ring and M a non-zero finitely generated left D-module. Then, $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ is either 0 or an i-pure left D-module.

Using Theorem 4.1 and (43), $\mathrm{coker}\,\gamma_{32}$ is a 2-pure left D-module, $\mathrm{coker}\,\gamma_{21}$ is a 1-pure left D-module and $M/t(M)$ is a 0-pure left D-module. Moreover, if R_3 has full row rank, namely $\ker_D(.R_3) = 0$, then $N_{33} \cong \mathrm{ext}_D^3(M, D)$, and thus $\mathrm{ext}_D^3(N_{33}, D) \cong \mathrm{ext}_D^3(\mathrm{ext}_D^3(M, D), D)$ is a 3-pure left D-module. Using the notations of (44), we then note that $M_3 \cong \mathrm{ext}_D^3(\mathrm{ext}_D^3(M, D), D)$ is a 3-pure left D-module, $M_2/M_3 \cong \mathrm{coker}\,\gamma_{32}$ is a 2-pure left D-module, $M_1/M_2 \cong \mathrm{coker}\,\gamma_{21}$ is a 1-pure left D-module and $M_0/M_1 \cong M/t(M)$ is a 0-pure left D-module, i.e., the successive quotients of the terms of the filtration $\{M_i\}_{i=0,\ldots,3}$ of M are pure left D-submodules of M. This filtration $\{M_i\}_{i=0,\ldots,3}$ of M is called the *purity filtration* of M [5], [9]. Now, if $\ker_D(.R_3) \neq 0$, i.e., $\ker_D(.R_3) = D^{1\times p_4} R_4$ for a non-trivial matrix $R_4 \in D^{p_4 \times p_3}$, then we can introduce the right D-module $N_{44} = D^{p_{44}}/(R_{44}\, D^{p_{34}})$, where $R_{44} = R_4$ and $p_{44} = p_4$, compute $\mathrm{ext}_D^4(N_{44}, D)$ and so on. Repeating the same procedure $\mathrm{gld}(D) = n$ times, we finally obtain a purity filtration $\{M_i\}_{i=0,\ldots,n}$ of M.

This explicit and rather elementary way for the computation of the purity filtration of a finitely presented left D-module M does not require sophisticated homological algebra techniques such as spectral sequences [3], [4], [5], associated cohomology [9], and Spencer cohomology [12], [13]. Efficient implementations of this new approach were recently done by the author in the PurityFiltration package, built upon OreModules [7], and, in collaboration with Barakat (University of Kaiserslautern), in the `AbelianSystems` package of the seminal `homalg` [2], [3] of GAP4 dedicated to homological algebra computations. The `homalg` package also includes the computation of the purity filtration based on spectral sequences. See [3] for a constructive study of spectral sequences of bicomplexes.

Finally, the results developed in this paper are used in [16] to explicitly determine a block-triangular linear system which is equivalent to $\ker_{\mathcal{F}}(R.)$. It will allow us to compute a Monge parametrization of the linear system $\ker_{\mathcal{F}}(R.)$ by integrating in cascade inhomogeneous linear systems defining equidimensional homogeneous linear systems.

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